DUAL INTEGRAL EQUATIONS ARISING IN LIMITING GRADIENT FILTRATION PROBLEMS

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A number of plane problems of filtration with a limiting gradient can be reduced [1] to obtaining the stream function ψ as a solution of the first boundary value problem for the equation

$$w(w+\lambda)\frac{\partial^2\psi}{\partial w^2}+(w-\lambda)\frac{\partial\psi}{\partial w}+\frac{\partial^2\psi}{\partial\theta^2}=0 \tag{0.1}$$

in the semi-infinite strip $0 \leqslant w < \infty$, $0 \leqslant \theta \leqslant \theta_0$ with a cut along the line $\theta = \theta_1 < \theta_0$ extending either from the point $w = a$ to $w = 0$ (Problem A), or from the point $w = a$ to $w = \infty$ (Problem B).

Below (Sect. 1) we reduce these problems to dual integral equations whose kernels contain hypergeometric functions $F(s, -u) = F(2+is, 2-is, 3, -u)$ dependent on the argument $u = w/\lambda$ and a single parameter s. The hypergeometric functions are expressed in terms of the associated Legendre functions. This enables us to write their integral representations in the form of Fourier sine and cosine transforms and then, following the work of Rukhovets and Ufliand [2] (see also [3]), reduce the dual equations to the Fredholm equation of the second kind (Sect. 2). This equation can be solved effectively when the values of a are small ($a_0 = a/\lambda \ll 1$) and this enables us to obtain those basic characteristic features of the filtration flow, which are of some interest to us. As an example, we establish in Sects. 3 and 4 the position and size of stagnation zones which arise when the source is situated near an impermeable boundary (this corresponds to Problem A), and near the rectilinear delivery contour (a particular case of Problem B).

1. We consider the following two problems.

P r o b l e m A. To find a solution of Eq. (0.1) in the semi-infinite strip $0 \leqslant w < \infty$, $0 \leqslant \theta \leqslant \theta_0$ with a cut $\theta = \theta_1$, $0 \leqslant w \leqslant a$, assuming specified values on the boundary of the strip and on the cut.

P r o b l e m B. To find a solution of Eq. (0.1) in the semi-infinite strip $0 \leqslant w < \infty$, $0 \leqslant \theta \leqslant \theta_0$ with a cut $\theta = \theta_1$, $a \leqslant w < \infty$ assuming specified values on the boundary strip and on the cut.

Let us introduce a new independent variable $u = w/\lambda$ into (0.1) and seek ψ as the function of u and θ. We set

$$\psi(u, \theta) = \Psi(u, \theta) + \psi_1(u, \theta) \tag{1.1}$$

where $\psi_1(u, \theta)$ is the solution of (0.1) in the semistrip $0 \leqslant w = \lambda u < \infty$, $0 \leqslant \theta \leqslant \theta_0$ without a cut, assuming the same values as $\psi(u, \theta)$ at the boundaries $\theta = 0$, $\theta = \theta_0$ and $w = 0$. The function $\psi_1(u, \theta)$ can readily be found.

Let the boundary conditions be given by

$$\psi_1(u, 0) = f_1(u), \ \psi_1(u, \theta_0) = f_2(u), \quad \psi_1(0, \theta) = \varphi(\theta) \tag{1.2}$$

We shall effect the integral transformation with respect to the variable u, assuming [4]

$$\psi_1^*(s, \theta) = \int_0^\infty (1+u) F(s, -u)\, \psi_1(u, \theta)\, du \tag{1.3}$$

(cont.)

$$\psi_1(u,\theta) = {}^1/_2u^2 \int_0^\infty s^3(1+s^2)\,\mathrm{cth}\,\pi s F(s,-u)\,\psi_1^*(s,\theta)\,ds$$

$$F(s,-u) \equiv F(2+is,\ 2-is,\ 3,\ -u)$$

Then $\psi^*_1(s,\theta)$ will be given by

$$\frac{d^2\psi_1^*(s,\theta)}{d\theta^2} - s^2\psi_1^*(s,\theta) = -2\varphi(\theta) \tag{1.4}$$

$$\psi_1^*(s,\theta) = f_1^*(s), \qquad \psi_1^*(s,\theta_0) = f_2^*(s)$$

$$f_{1,2}^*(s) = \int_0^\infty (1+u)\,F(s,-u)\,f_{1,2}(u)\,du \tag{1.5}$$

Having found $\psi^*_1(s,\theta)$, we can use (1.3) to define $\psi_1(s,\theta)$.

We can therefore assume that the function $\psi_1(s,\theta)$ is known and consider at once Problems A and B with zero boundary conditions at $\theta=0$, $\theta=\theta_0$, $u=0$, without loss of generality.

Let us now consider Problem A with the following boundary conditions:

$$\psi(u,0) = \psi(0,\theta) = \psi(u,\theta_0) = 0$$

$$\psi(u,\theta_1 \pm 0) = f_\pm(u), \qquad 0 < u < a_0 = a/\lambda \tag{1.6}$$

We denote the trasform of the required solution (i.e. the result of applying the integral transformation (1.3)), by $\psi^*(s,\theta)$. When $0 \leqslant \theta \leqslant \theta_1$ we have, with the boundary condition at $\theta=0$ and $u=0$ taken into account

$$\psi^*(s,\theta) = A(s)\,\mathrm{sh}\,s\theta \tag{1.7}$$

and for $\theta_1 < \theta < \theta_0$ we have similarly

$$\psi^*(s,\theta) = B(s)\,\mathrm{sh}\,s(\theta_0-\theta) \tag{1.8}$$

When $\theta=\theta_1$ we have by virtue of (1.6) and (1.3)

$$\psi^*(s,\theta+0) - \psi^*(s,\theta-0) = \int_0^{a_0} (1+u)\,F(s,-u)\,[f_+(u) - f_-(u)]\,du = \sigma(s)$$

from which we obtain

$$B(s) = [\sigma(s) + A(s)\,\mathrm{sh}\,s\theta_1]/\,\mathrm{sh}\,s\theta_2, \qquad \theta_2 = \theta_0 - \theta_1 \tag{1.9}$$

To satisfy all conditions of the problem we require that the derivative $\partial\psi/\partial\theta$ is continuous when $\theta=\theta_1$ and $u > a_0$, and that the condition $\psi(u,\theta_1-0) = f_-(u)$, $u < a_0$ holds. With (1.3) and (1.9) taken into account, these conditions yield

$$\int_0^\infty s^3(1+s^2)\,\mathrm{cth}\,\pi s\,F(s,-u)\,C(s)\,ds = 0 \qquad (a_0 < u < \infty) \tag{1.10}$$

$$\int_0^\infty \frac{s^2(1+s^2)}{\mathrm{th}\,\pi s}\,\frac{\mathrm{sh}\,s\theta_1\,\mathrm{sh}\,s\theta_2}{\mathrm{sh}\,s\theta_0}\,C(s)\,F(s,-u)\,ds = h(u) \qquad (0 < u < a_0)$$

$$C(s) = s\,[A(s)\,\mathrm{sh}\,s\theta_0 + \sigma(s)\,\mathrm{ch}\,s\theta_2]\,/\,\mathrm{sh}\,s\theta_2 \tag{1.11}$$

$$h(u) = -\frac{2f_-(u)}{u^2} + \int_0^\infty s^3(1+s^2)\,\frac{\mathrm{ch}\,s\theta_2\,\mathrm{sh}\,s\theta_1}{\mathrm{sh}\,s\theta_0\,\mathrm{th}\,\pi s}\,F(s,-u)\,\sigma(s)\,ds \tag{1.12}$$

Thus, Problem A is reduced to solving the dual equations (1.10) in order to define $C(s)$.

Let us now consider Problem B with the following conditions:

$$\psi(u,0)=\psi(0,\theta)=\psi(u,\theta_0)=0,\quad \psi(u,\theta_1\pm 0)=F_\pm(u),\quad u>a_0 \tag{1.13}$$

We can, as before, write the transform $\psi^*(s,\theta)$ in the form given by (1.7) and (1.8) where $B(s)$ and $A(s)$ are related by (1.9) in which $\sigma(s)$ has been relaced with

$$\sigma_1(s)=z_+(s)-z_-(s),\quad z_\pm(s)=\int_{a_0}^{\infty}(1+u)F_\pm(u)F(s,-u)\,du \tag{1.14}$$

Satisfying now the continuity condition for $\partial\psi/\partial\theta$ when $\theta=\theta_1$ and $u<a_0$ and the condition $\psi(u,\theta_1-0)=F_-(u), u>a_0$, we obtain the following dual equations:

$$\int_0^\infty s(1+s^2)\operatorname{cth}\pi s\, C(s)F(s,-u)\,ds=0\qquad (a_0<u<\infty) \tag{1.15}$$

$$\int_0^\infty \frac{s^2(1+s^2)\operatorname{sh}s\theta_0}{\operatorname{sh}s\theta_1\operatorname{sh}s\theta_2\operatorname{th}\pi s}C(s)F(s,-u)\,ds=h_2(u)\qquad(0\leqslant u\leqslant a_0)$$

$$C(s)=s^2\left[A(s)\operatorname{sh}s\theta_1-\int_{a_0}^{\infty}(1+u)F_-(u)F(s,-u)\,du\right] \tag{1.16}$$

$$h_2(u)=\int_0^\infty\frac{s^4(1+s^2)}{\operatorname{th}\pi s}\left[\frac{\sigma_1(s)}{\operatorname{th}s\theta_2}+\frac{z_-(s)\operatorname{sh}s\theta_0}{\operatorname{sh}s\theta_1\operatorname{sh}s\theta_2}\right]F(s,-u)\,ds \tag{1.17}$$

In what follows we shall find it convenient to integrate the dual equations (1.10) and (1.15) with respect to u, from u to ∞. Taking into account the relation

$$\int_u^\infty F(2+is,2-is,3,-u)\,du=\frac{2}{1+s^2}F(1+is,1-is,2,-u) \tag{1.18}$$

which follows from the differentiation formulas of the hypergeometric functions [5 and 6], we obtain

$$\int_0^\infty s^3\operatorname{cth}\pi s\,F_1(s,-u)\,C(s)\,ds=0\qquad(a_0<u<\infty) \tag{1.19}$$

$$\int_0^\infty s^2F_1(s,-u)\,C(s)\,\frac{\operatorname{sh}s\theta_1\operatorname{sh}s\theta_2}{\operatorname{th}\pi s\operatorname{sh}s\theta_0}\,ds=-\int^u\frac{f_-(u)}{u^2}\,du+$$

$$+\int_0^\infty s^3\frac{\operatorname{ch}s\theta_2\operatorname{sh}s\theta_1}{\operatorname{sh}s\theta_0\operatorname{th}\pi s}F_1(s,-u)\,\sigma(s)\,ds\equiv H_1(u)+r_1$$

for Problem A and

$$\int_0^\infty s\operatorname{cth}\pi s\,F_1(s,-u)\,C(s)\,ds=0\qquad(u>a_0) \tag{1.20}$$

$$\int_0^\infty s^2\operatorname{cth}\pi s\,C(s)F_1(s,-u)\frac{\operatorname{sh}s\theta_0}{\operatorname{sh}s\theta_1\operatorname{sh}s\theta_2}\,ds=\frac{1}{2}\int_u h_2(u)\,du\equiv H_2(u)+r_2\qquad(0<u<a_0)$$

for Problem B. Here

$$F_1(s,-u)=F(1+is,1-is,2,-u),\quad F_0(s,-a_0)=F(is,-is,2,-a_0) \tag{1.21}$$

The constants of integration r_1 and r_2 are defined from the solution requirements.

2. 1°. To reduce the dual equations of the form (1.19) and (1.20) to the Fredholm equations of the second kind, we shall use the method applied in [2 and 3] for solving dual equations containing associated Legendre functions. This method makes use of the integral representations of the Legendre functions in the form of sine and cosine transforms. Corresponding representations can also be obtained for the hypergeometric functions appearing in the present paper. In particular, for $F_1(s, -u)$ we can obtain

$$F_1(s, -u) = \frac{2}{\pi s u}\int_0^{\beta} \frac{\sin 2\eta s \operatorname{sh}\eta \, d\eta}{\sqrt{\operatorname{sh}^2\beta - \operatorname{sh}^2\eta}}, \quad u = \operatorname{sh}^2\beta \tag{2.1}$$

$$F_1(s, -u) = -(\pi s u)^{-1} \operatorname{th} \pi s \,[R_+(s, u) + R_-(s, u)] = \tag{2.2}$$

$$= \frac{2 \operatorname{th} \pi s}{\pi s u}\left[\frac{\sin 2\beta s}{2s} - \int_{\beta}^{\infty} \frac{\cos 2\eta s \,(\operatorname{sh}\eta - \sqrt{\operatorname{sh}^2\eta - \operatorname{sh}^2\beta})\, d\eta}{\sqrt{\operatorname{sh}^2\eta - \operatorname{sh}^2\beta}}\right]$$

Functions

$$R_{\pm}(s, u) = \pm \frac{e^{\mp 2is\beta}}{2is} + \int_{\beta}^{\infty} e^{\mp 2is\eta} \frac{\operatorname{sh}\eta - \sqrt{\operatorname{sh}^2\eta - \operatorname{sh}^2\beta}}{\sqrt{\operatorname{sh}^2\eta - \operatorname{sh}^2\beta}}\, d\eta \tag{2.3}$$

are analytic in the half-planes $\operatorname{Im} s < 1$ and $\operatorname{Im} s > -1$ respectively.

The above formulas are obtained by using analytic continuation to change the argument of the hypergeometric function to $1/u$ and utilizing their representation in the form of an Euler integral. They can also be obtained directly, although more tediously, from the integral representations of the Legendre functions [5 and 6] taking into account the relation

$$F(n + is,\ n - is,\ c,\ -u) = (-1)^n \frac{\Gamma(c)\Gamma(is)\Gamma(-is)}{2\Gamma(n + is)\Gamma(n - is)} \times$$

$$\times \frac{d^n}{du^n}\left[\left(\frac{u}{u+1}\right)^{1/2(n-c+1)} \left(P_{-is}^{n-c+1}(1 + 2u) + P_{is}^{n-c+1}(1 + 2u)\right)\right] \tag{2.4}$$

connecting the hypergeometric functions appearing in this paper, with the associated Legendre functions.

2°. We shall seek a solution of (1.19) in the form

$$C(s) = s^{-1}\int_0^{2\beta_0} \varphi(\tau) \sin s\tau \, d\tau, \qquad \operatorname{sh}^2\beta_0 = a_0 \tag{2.5}$$

where $\varphi(\tau)$ is a function continuously differentiable on the segment $(0, 2\beta_0)$. In addition, we stipulate that the first equation of (1.19) be an identity. Indeed, using (2.2) we can write the left side of the equation (1.19) as

$$I(u) = \frac{1}{2\pi u}\int_{-\infty}^{\infty} s^{\circ} R_+(s, u)\, C(s)\, ds$$

In the lower half-plane the function R_+, as follows from (2.3), decreases with $s \to \infty$ at least as fast as $s^{-1/2} e^{2i\beta s}$, the function $C(s)$ is, by virtue of (2.5), an entire analytic function and increases with $s \to \infty$ not faster than $s^{-2} e^{2i\beta_0 s}$. Since we also have $\beta > \beta_0$ when $u > a_0$ we have as the result $I(u) = 0$. Let us set

$$2 \operatorname{cth} \pi s \operatorname{sh} s\theta_1 \operatorname{sh} s\theta_2 / \operatorname{sh}\theta_0 = 1 + \varepsilon_1(s) \tag{2.6}$$

Inserting the expressions (2.1), (2.5) and (2.6) into the second equation of (1.19), we transform it into the form

$$\frac{1}{\pi u}\int_0^\infty [1+\varepsilon_1(s)]\int_0^\infty \frac{\sin 2\eta\, s\, \mathrm{sh}\,\eta\, d\eta}{\sqrt{\mathrm{sh}^2\beta-\mathrm{sh}^2\eta}}\int_0^{2\beta_0}\varphi(\tau)\sin s\tau\, d\tau\, ds = H_1(u)+r_1 \qquad (2.7)$$

Let us denote the cosine transform of $\varepsilon_1(s)$ by

$$E_1(\alpha)=\int_0^\infty \varepsilon_1(s)\cos\alpha s\, ds \qquad (2.8)$$

It can easily be shown that the product

$$(1+\varepsilon_1(s))\int_0^{2\beta_0}\varphi(\tau)\sin s\tau\, d\tau$$

is the sine transform of the quantity

$$X(\tau)=\varphi(\tau)+\pi^{-1}\int_0^{2\beta_1}\varphi(\alpha)\,[E_1(\tau-\alpha)-E_1(\tau+\alpha)]\,d\alpha \qquad (2.9)$$

The second cofactor in the integrand of (2.7) represents, in turn, the sine transform of a function equal to $\frac{1}{2}\mathrm{sh}\frac{1}{2}\eta/\sqrt{\mathrm{sh}^2\beta-\mathrm{sh}^2\frac{1}{2}\eta}$ when $\eta<2\beta$, and equal to zero when $\eta>2\beta$. We can therefore apply the convolution theorem for the sine transforms [7] to (2.7), to obtain

$$\int_0^{2\beta} X(\tau)\frac{\mathrm{sh}\frac{1}{2}\tau\, d\tau}{\sqrt{\mathrm{sh}^2\beta-\mathrm{sh}^2\frac{1}{2}\tau}} = 4uH_1(u)+4ur_1 \qquad (2.10)$$

Setting now $\mathrm{ch}\frac{1}{2}\tau=\xi$ we transform (2.10) into the Abel integral equation [8] whose solution is

$$X(\tau)=\frac{4}{\pi\,\mathrm{sh}\frac{1}{2}\tau}\frac{d}{d\tau}\int_0^{\mathrm{sh}^2\frac{1}{2}\tau}\frac{uH_1(u)+ur_1}{\sqrt{\mathrm{sh}^2\frac{1}{2}\tau-u}}\,du \qquad (2.11)$$

3. 1°. We apply the results obtained in the previous Sections to problems which actually arise during the investigations of plane filtration flows following the law of filtration with a limiting gradient. Study of a flow generated by an aggregate of several, equal intensity sources, situated at the vertices of a regular n-sided polygon leads [1] to the following particular case of Problem A:

$$\psi(u,0)=0,\quad \psi(0,\theta)=0\quad(0\leqslant\theta\leqslant\theta_1),\quad \psi(u,\theta_1)=0\quad(0\leqslant u\leqslant a_0) \qquad (3.1)$$
$$\psi(0,\theta)=Q(\theta-\theta_1)/\theta_2\ (\theta_1\leqslant\theta\leqslant\theta_0)$$
$$\psi(u,\theta_0)=Q\quad(\theta_0=\pi,\ \theta_1=\pi(n-1)/n)$$

Here the problem of determination of the boundary of the stagnation zone appears to be the most important one. Within the symmetry element of the flow, this boundary depends on the position of the tip of the stagnation zone relative to the source

$$z_0=x_0+iy_0=\frac{1}{\lambda}\int_0^\infty\frac{\partial\psi(u,0)}{\partial\theta}\frac{du}{u^2} \qquad (3.2)$$

and on the form of the boundary arc

$$z(\theta)-z_0=x(\theta)+iy(\theta)-z_0=\frac{1}{\lambda}\int_0^\theta\lim_{u\to 0}\left(\frac{1}{u}\frac{\partial\psi(u,\theta)}{\partial u}\right)e^{i\theta}d\theta,\quad 0\leqslant\theta\leqslant\theta_1 \qquad (3.3)$$

Here x and y are the coordinates in the plane of the flow, their origin coinciding with one of the sources and the axis directed towards the center of symmetry of the flow.

Setting $n = 2$ we obtain a flow due to two sources of equal intensity.

An important case of the problem B arises in the study of a flow due to n sources of intensity $2Q$ distributed at the vertices of a regular n-sided polygon and a sink of intencity $2Qn$ situated at its center ($n = 1, 2, \ldots$). We have

$$\theta_1 = \pi, \quad \theta_0 = \pi(1+n)/n, \quad \psi_2(u, \theta) = 0, \quad F_+(u) = F_-(u) = Q = \text{const} \tag{3.4}$$

In this case an outer stagnation zone occurs, its boundary is defined by the relations (3.2) and (3.3). We also have $0 \leqslant \theta \leqslant \theta_0$, the coordinate origin coincides with one of the sources and the sink lies on the negative part of the x-axis.

When $n = 1$, the problem corresponds to an equal intensity source-sink pair.

In all cases the quantity a_0 is assumed given and the radius of the source aggregate is to be determined (i.e. we essentially solve the converse problem).

The above flow as well as some other flows leading to Problems A and B are all considered in [1].

2°. It can easily be shown that the solution $\psi_1(u, \theta)$ (see Sect. 1) with the boundary conditions (3.1), has the form ($0 < \theta < \theta_1$)

$$\psi_1(u,\theta) = \frac{u^2 Q}{\theta_2}\int_0^\infty \frac{1+s^2}{\operatorname{th}\pi s}\,\frac{\operatorname{sh} s\theta_2 \operatorname{sh} s\theta}{\operatorname{sh} s\theta_0}\,F(s, -u)\,ds \tag{3.5}$$

so that

$$f_+(u) = f_-(u) = -\psi_1(u, \theta_1) \quad (u \leqslant a_0), \ \sigma(s) = 0 \tag{3.6}$$

and in accordance with (1.12), (1.21) and (1.18) we have

$$H_1(u) = -\frac{2Q}{\theta_2}\int_0^\infty \frac{F_1(s,-u)}{\operatorname{th}\pi s}\,\frac{\operatorname{sh} s\theta_1 \operatorname{sh} s\theta_2}{\operatorname{sh} s\theta_0}\,ds \tag{3.7}$$

Replacing the function $F_1(s, -u)$ with its integral representation (2.1) and inserting the resulting expression into (2.11) we obtain after some manipulations

$$X_1(\tau) = -\frac{2Q}{\theta_2}\left[1 + \frac{2}{\pi}\int_0^\infty \varepsilon_1(s)\,\frac{\sin s\tau}{s}\,ds\right] + \frac{4}{\pi} r_1 \operatorname{sh}\tau \tag{3.8}$$

where $\varepsilon_1(s)$ is defined by (2.6). Further, the kernel of (2.9), with (2.8) and (2.6) taken into account, admits the representation in the form of a uniformly converging series

$$E_1(\tau-\alpha) - E_1(\tau+\alpha) = 2\tau\alpha\, E_{11} - {}^1/_3\,\tau\alpha\,(\tau^2+\alpha^2)\,E_{12} + \ldots \tag{3.9}$$

$$E_{1k} = \int_0^\infty s^{2k}\varepsilon_1(s)\,ds$$

Taking into account (3.8) and (3.9) we find from (2.9), that the solution $\varphi(\tau)$ can be written as

$$\varphi(\tau) = -2Q/\theta_2 + \varphi_1(\tau) \tag{3.10}$$

where $\varphi_1(\tau)$ is a function which can be expanded into a series in odd powers of τ, for any value of r_1.

3°. The solution of the initial problem with $0 \leqslant \theta \leqslant \theta_1$ is given by

$$\psi(u,\theta) = \Psi(u,\theta) + \psi_1(u,\theta) = \tag{3.11}$$

$$= \frac{1}{2}u^2\int_0^\infty \left[s^2 C(s) + \frac{2Q}{\theta_2}\right](1+s^2)\,\frac{\operatorname{sh} s\theta \operatorname{sh} s\theta_2}{\operatorname{th}\pi s \operatorname{sh} s\theta_0}\,F(s, -u)\,ds$$

where $C(s)$ is defined by (2.5).

When $u < a_0$ and $\theta = \theta_1$, we have $\psi(u, \theta_1) \equiv 0$ and the derivative $\partial\psi / \partial\theta$ must be an integrable function of u. Taking into account (1.18), (2.2) and (3.10) we have

$$\frac{\partial\psi(u, \theta_1)}{\partial\theta} = {}^1/_4 u^2 \int_0^\infty \frac{s^3(1+s^2)F(s, -u)}{\operatorname{th}\pi s}\left[C(s) + \frac{2Q}{s^2\theta_2}\right]\left(1 + \frac{\varepsilon_1(s)}{\operatorname{th}\theta_1 s}\right)ds =$$

$$= -u^2 \frac{1}{du}\int_0^\infty \frac{\operatorname{ch} s\theta_1 \operatorname{sh} s\theta_2}{\pi u \operatorname{sh} s\theta_0}\left[\int_0^{2\beta_0} \varphi'(\tau)\cos s\tau d\tau - \varphi(2\beta_0)\cos 2\beta_0 s\right] \times$$

$$\times\left[\frac{\sin 2\beta s}{s} - 2\int_\beta^\infty \frac{\cos 2\eta s\,[\operatorname{sh}\eta - \sqrt{\operatorname{sh}^2\eta - \operatorname{sh}^2\beta}]}{\sqrt{\operatorname{sh}^2\eta - \operatorname{sh}^2\beta}}\,d\eta\right]ds \qquad (3.12)$$

Behavior of the solution near the point $u = a_0$ ($\beta = \beta_0$) can easily be determined by considering the integral (3.12) as an inverse Fourier cosine transform. When $s \to \infty$, the expression within the square brackets is of the order of

$$s^{-1}\sin 2\beta s + Cs^{-1/2}e^{2i\beta s}$$

therefore the asymptotic form of the integral as $\beta \to \beta_0$ is given by

$$C_1(\beta_0 - \beta)^{-1/2}\varphi(2\beta_0) + C_2\varphi(2\beta_0) + O(1)$$

Hence the integrability of $\partial\psi / \partial\theta$ implies that

$$\varphi(2\beta_0) = 0 \qquad (3.13)$$

which can be used to determine the constant r_1. Finally, inserting (3.11) into the expression (3.3) for the coordinates of the boundary of the stagnation zone and taking into account (3.10) and (3.12), we obtain

$$\lambda[z(\theta) - z_0] = \left[e^{i\theta}\int_0^\infty \frac{s\operatorname{ch} s\theta - i\operatorname{sh} s\theta}{s\operatorname{ch} s\theta_0}\operatorname{sh} s\theta_2 \operatorname{cth}\pi s \times\right.$$

$$\left.\times\left(\varphi'(2\beta_0)\sin 2s\beta_0 - \int_0^{2\beta_0}\varphi''(\tau)\sin s\tau d\tau\right)ds\right]_{\theta=0}^{\theta=\theta_0} \qquad (3.14)$$

Expression (3.11) remains finite when $0 \leqslant \theta \leqslant \theta_1$.

4°. Let us now obtain an approximate solution of the problem for small values of β_0. Using (3.8), (3.9) and (3.13) we obtain from the integral equation (2.9) the following expression

$$\varphi(\tau) = \frac{2Q}{\theta_2}\left[-1 + \left(\frac{1}{2\beta_0} - \frac{1}{3}\beta_0 - \frac{4}{3\pi}E\beta_0^2\right)\tau + \frac{\tau^3}{3\pi}\left(\frac{\pi}{4\beta_0} + E\right) + O(\beta_0^4)\right]$$

$$E = E_{11} + E_{12} \qquad (3.15)$$

Insertion of $\varphi(\tau)$ into (3.14) yields

$$\lambda[z(\theta) - z_0] = \frac{2Q}{\theta_2}\int_0^\infty \frac{e^{i\theta}(s\operatorname{ch} s\theta - i\operatorname{sh} s\theta) - s}{\operatorname{sh} s\theta_0}\operatorname{sh} s\theta_2 \operatorname{cth}\pi s \times \qquad (3.16)$$

$$\times\left[\frac{\sin 2\beta_0 s}{s}\left(\frac{1}{2\beta_0} + {}^2/_3\beta_0 + \frac{8}{3\pi}\beta_0^2 E + O(\beta_0^3)\right) - \left(\frac{1}{2\beta_0} + \frac{2}{\pi}E\right)\int_0^{2\beta_0}\tau\frac{\sin s\tau}{s}\,d\tau\right]ds$$

When $s \to \infty$, the expression in front of the square bracket in (3.16) behaves as ${}^1/_2(s - i)\,e^{s(\theta-\theta_1)+i\theta}$. Therefore for small β_0 we have

$$\lambda\,[z(\theta)-z_0]=\frac{2Q}{\theta_2}\,e^{i\theta}\left[\int\limits_0^\infty\left(\frac{s\operatorname{ch}s\theta\operatorname{sh}s\theta_2}{\operatorname{th}\pi s\operatorname{sh}s\theta_0}-\frac{1}{2}\,se^{s(\theta-\theta_1)}\right)ds+\frac{1}{2\,[4\beta_0^2+(\theta_1-\theta)^2]}\times\right.$$
$$\times\left(1+{}^4/_3\beta_0^2+\frac{16}{3\pi}\beta_0^3E\right)-{}^1/_2\left(1+\frac{4\beta_0}{\pi}E\right)\left(2-\frac{\theta_1-\theta}{2\beta_0}\operatorname{arctg}\frac{2\beta_0}{\theta_1-\theta}\right)-$$
$$-i\left\{\left(\frac{1}{4\beta_0}+{}^1/_3\beta_0\right)\operatorname{arctg}\frac{2\beta_0}{\theta_1-\theta}-\frac{1}{8\beta_0}\Big[(4\beta_0^2+(\theta_1-\theta)^2)\operatorname{arctg}\frac{2\beta_0}{\theta_1-\theta}-\right.$$
$$\left.\left.-2\beta_0(\theta_1-\theta)\Big]+\int\limits_0^\infty\left[\frac{\operatorname{sh}s\theta\operatorname{sh}s\theta_2\operatorname{cth}\pi s}{\operatorname{sh}s\theta_0}-{}^1/_2e^{-s(\theta_1-\theta)}\right]ds\right\}\right]-$$
$$-\frac{2Q}{\theta_2}\int\limits_0^\infty\frac{s\operatorname{sh}s\theta_2\operatorname{cth}\pi s}{\operatorname{sh}s\theta_0}\,ds \tag{3.17}$$

To determine the location of the stagnation zone completely, we require the value of z_0. Using (3.11), ((1.18) and (3.13) we have from (3.2)

$$\lambda z_0=\int\limits_0^\infty\frac{1}{u^2}\,\frac{\partial\psi(u,0)}{\partial\theta}\,du=\int\limits_0^\infty\frac{s\operatorname{sh}s\theta_2\operatorname{cth}\pi s}{\operatorname{sh}s\theta_0}\int\limits_0^{2\beta_0}\varphi'(\tau)\cos s\tau\,d\tau\,ds=$$
$$=\frac{2Q}{\theta_2}\int\limits_0^\infty\frac{s\operatorname{sh}s\theta_2}{\operatorname{sh}s\theta_0}\operatorname{cth}\pi s\,ds+O(\beta_0^2) \tag{3.18}$$

5°. In [1] we have constructed the limit solution of Problem A $(\alpha_0=\beta_0=0)$ which enabled us to obtain lower estimates for the stagnation zone sizes. However in that case the boundary of the stagnation zone consisted of branches extending to infinity and the solution did not therefore possess any inherent value. The solution given above for small β_0 approximates the limit solution at all θ such that $\theta_1-\theta\gg\beta_0$ and it is only in the vicinity of $\theta=\theta_1$ that the boundary of the stagnation zone begins to deviate appreciably from the limit boundary, for the reason of absence of the infinite branches.

As an example, we shall consider a flow due to a pair of sources of equal strength (strength $q=2Q$).

In this case we have $\theta_0=2\theta_1=\pi$. The limit solution was constructed in [1]. We also have

$$E_{10}=3E_{11}=-\frac{1}{\pi},\qquad E=-\frac{4}{3\pi}$$

$$\lambda\,(z(\theta)-z_0)=\frac{Q}{\pi}\,e^{i\theta}\left\{\frac{1}{\cos^2\theta}+\frac{2}{\pi\cos\theta}+\frac{2\theta}{\pi}\,\frac{\sin\theta}{\cos^2\theta}-\frac{8}{(\pi-2\theta)^2}+\right.$$
$$+\left(2+{}^8/_3\beta_0^2-\frac{128}{9\pi^2}\beta_0^3\right)\frac{1}{4\beta_0^2+({}^1/_2\pi-\theta)^2}+\left(1-\frac{16\beta_0}{3\pi^2}\right)\left(-2+\right.$$
$$\left.+\frac{\pi-2\theta}{2\beta_0}\operatorname{arctg}\frac{4\beta_0}{\pi-2\theta}\right)-i\left[\left(\frac{1}{\beta_0}-{}^2/_3\beta_0-\frac{(\pi-2\theta)^2}{8\beta_0}\right)\operatorname{arctg}\frac{4\beta_0}{\pi-2\theta}+\right.$$
$$\left.\left.+\frac{\pi}{2}-\theta+\operatorname{tg}\theta+\frac{2\theta}{\pi\cos\theta}-\frac{4}{\pi-2\theta}\right]\right\} \tag{3.19}$$

$$\lambda z_0=\frac{4Q}{\pi}\int\limits_0^\infty\frac{s\operatorname{sh}{}^1/_2s\pi\operatorname{ch}s\pi}{\operatorname{sh}^2s\pi}\,ds=\frac{2+\pi}{\pi^2}\,Q=0.521Q \tag{3.20}$$

For the tip of the stagnation zone $(\theta={}^1/_2\pi)$ we have

$$\lambda z\,({}^1/_2\pi)=\frac{Q}{2\beta_0}-\frac{2Q}{\pi^2}+i\,\frac{Q}{\pi}\left(\frac{1}{2\beta_0^2}+\frac{5}{6}\right) \tag{3.21}$$

Figure 1 depicts the boundaries of the stagnation zones for various values of $\beta_0 (a_0 = \mathrm{sh}^2\beta_0)$.

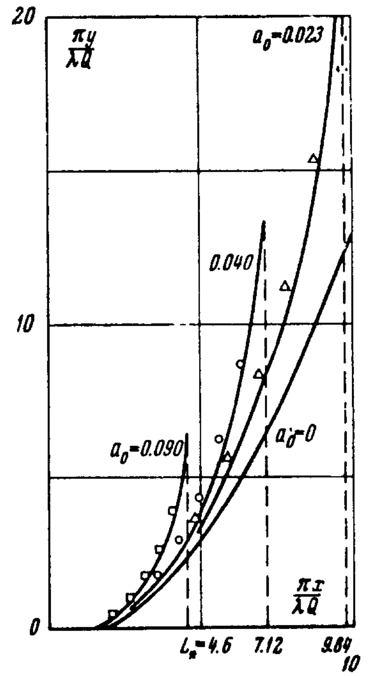

Fig. 1

We can write the equation of the boundary of the stagnation zone in the form

$$\left(\frac{L-x}{L-x(0)}\right)^{2/3} + \left(\frac{y}{y(^1/_2\pi)}\right)^{2/3} = 1 \qquad (3.22)$$

(points on Fig. 1) where $L = x(^1/_2\pi)$ and $y\,(^1/_2\pi)$ are given by (3.21). This equation is sufficiently accurate for practical purposes. It has the shape of an astroid elongated in the y-direction and its total area is given by

$$S = {}^3/_{16}\pi\,(L - x(0))\,y(^1/_2\pi) \approx$$

$$\approx {}^3/_8 \frac{\lambda L^3}{Q}\left(1 + \frac{2Q}{\pi\lambda L}\right)^2 \left(1 - \frac{\pi + 2}{\pi\lambda L} Q\right)$$

$$\frac{\lambda L}{Q} \gg 1 \qquad (3.23)$$

We remind that here $Q = {}^1/_2 q$ where q denotes the output of a single pore per unit capacity of the layer.

4. 1°. We shall now consider the case of Problem B given in Sect. 3, 1°, and defined by the conditions (3.4), in more detail. Here from (1.14) we obtain

$$z_+ = z_- = 2Qs^{-2}F\,(is,\,-is,\,2,\,-a_0),\ \sigma_1 \equiv 0 \qquad (4.1)$$

and in accordance with (1.17), (1.18) and (1.21),

$$H_2(u) = -2Q\int_0^\infty s^3\,\frac{\mathrm{sh}\,s\theta_0}{\mathrm{th}\,\pi s}\,\frac{F_1(s,\,-u)\,F_0(s,\,-a_0)}{\mathrm{sh}\,s\theta_1\,\mathrm{sh}\,s\theta_2}\,ds \qquad (4.2)$$

Using (2.4) and the well known integral representations of the Legendre functions we can obtain the following expression for the function $F_0\,(s,\,-a_0) \equiv F(is,\,-is,\,2,\,-a_0)$:

$$F\,(is,\,-is,\,2,\,-a_0) = \frac{1}{\pi(1+s^2)}\int_0^{2\beta_0}\frac{s\,\mathrm{cth}^2\,\beta_0\,\mathrm{sh}\,{}^1/_2\tau\,\sin s\tau + \mathrm{ch}\,{}^1/_2\tau\cos s\tau}{(\mathrm{sh}^2\,\beta_0 - \mathrm{sh}^2\,{}^1/_2\tau)^{1/2}}\,d\tau =$$

$$= \frac{1}{\pi s}\int_0^{2\beta_0}\frac{\mathrm{sh}\,{}^1/_2\tau\,(\mathrm{ch}\,\tau - \mathrm{sh}^2\,\beta_0)\sin s\tau}{\mathrm{sh}^2\,\beta_0\,(\mathrm{sh}^2\,\beta_0 - \mathrm{sh}^2\,{}^1/_2\tau)^{1/2}}\,d\tau,\ \ \mathrm{sh}^2\beta_0 = a_0 \qquad (4.3)$$

To solve Problem B we can use again the representation $C\,(s)$ in the form (2.5). We find however that the requirements imposed on $\varphi\,(\tau)$ become somewhat different since the first relation of (1.20) contains s, while (1.19) contains s^3. Repeating the arguments of Sect. 2, 2° we find, that in the present case we need the following necessary condition:

$$\int_0^{2\beta_0} \tau\varphi\,(\tau)\,d\tau = 0 \qquad (4.4)$$

in addition to the sufficient condition of integrability of $\varphi\,(\tau)$ on the segment $(0,\,2\beta_0)$.

Condition (4.4) will be used later to determine the constant r_2. The problem of determining $\varphi(\tau)$ again reduces to an integral equation of the form (2.9) in which $\varepsilon_1(s)$ is replaced by

$$\varepsilon_2(s) = {}^1/_2 \operatorname{cth} \pi s \operatorname{sh} s\theta_0 [\operatorname{sh} s\theta_1 \operatorname{sh} s\theta_2]^{-1} - 1 \tag{4.5}$$

and $H_1(u)$ in (2.11) by ${}^1/_4 H_2(u)$. Using the expressions (4.2) and (4.3) we find that the function $\varphi(\tau)$ can be written as

$$\varphi(\tau) = -\frac{2Q}{\pi} \frac{\operatorname{sh} {}^1/_2\tau (\operatorname{ch}\tau - \operatorname{sh}^2\beta_0)}{\operatorname{sh}^2\beta_0 (\operatorname{sh}^2\beta_0 - \operatorname{sh}^2 {}^1/_2\tau)^{1/2}} + r_2\zeta(\tau) \tag{4.6}$$

Here $\zeta(\tau)$ satisfies the equation

$$\zeta(\tau) + \frac{1}{\pi}\int_0^{2\beta_0} \zeta(\alpha) K(\alpha,\tau)\,d\alpha = \frac{1}{\pi}\operatorname{sh}\tau \tag{4.7}$$

where

$$K(\alpha,\tau) = 2\int_0^\infty \sin\alpha s \sin\tau s \left[\frac{\operatorname{cth}\pi s \operatorname{sh} s\theta_0}{2\operatorname{sh}\theta_1 s \operatorname{sh}\theta_2 s} - 1\right] ds \tag{4.8}$$

By (4.7) and (4.8) $\zeta(\tau)$ can be expanded into a series in odd powers of τ.

2°. Using the expressions (1.3), (1.7), (1.16), (2.5), (4.3) and (4.6), we can write the solution $\psi(u,\theta)$ for $0 \leqslant \theta \leqslant \theta_1$ as

$$\psi(u,\theta) = \frac{u^2}{2}\int_0^\infty \frac{s(1+s^2)\operatorname{sh} s\theta}{\operatorname{th}\pi s \operatorname{sh} s\theta_1}\left[\frac{1}{s}\int_0^{2\beta_0} \varphi(\tau)\sin s\tau\,d\tau + 2QF_0(s,-a_0)\right] F(s,-u)\,ds =$$

$$= \frac{r_2 u^2}{2}\int_0^\infty \frac{(1+s^2)\operatorname{sh} s\theta}{\operatorname{th}\pi s \operatorname{sh} s\theta_1}\int_0^{2\beta_0} \zeta(\tau)\sin s\tau\,d\tau\,ds \tag{4.9}$$

Coordinates of the boundary of the stagnation zone are given here by the expressions derived from (4.8), (3.2) and (3.3)

$$\lambda z_0 = r_2 \int_0^\infty s\frac{\operatorname{cth}\pi s}{\operatorname{sh} s\theta_1}\int_0^{2\beta_0} \zeta(\tau)\sin s\tau\,d\tau\,ds \tag{4.10}$$

$$\lambda[z(\theta) - z_0] = r_2 e^{i\theta}\left(\frac{\partial}{\partial\theta} - i\right)\int_0^\infty \frac{\operatorname{sh} s\theta \operatorname{cth}\pi s}{\operatorname{sh} s\theta_1}\int_0^{2\beta_0} \zeta(\tau)\sin s\tau\,d\tau\,ds \tag{4.11}$$

For small β_0 the above relations give

$$\lambda z_0 = 2Q\int_0^\infty \frac{s^2 \operatorname{cth}\pi s}{\operatorname{sh} s\theta_1}\,ds + 0(a_0) \tag{4.12}$$

$$\lambda[z(\theta) - z_0] \approx -\left\{2Qe^{i\theta}\left(i - \frac{\partial}{\partial\theta}\right)\int_0^\infty s\left[\frac{\operatorname{sh} s\theta \operatorname{cth}\pi s}{\operatorname{sh} s\theta_1} - e^{-s\xi}\right] ds - \right.$$

$$- ie^{i\theta}\frac{3Q}{4\beta_0^3}(1 - {}^2/_5\beta_0^2)\left[2\beta_0 - {}^1/_3\beta_0\xi^2 + {}^4/_9\beta_0^3 - \xi(1 - {}^1/_6\xi^2)\operatorname{arctg}\frac{2\beta_0}{\xi}\right] +$$

$$\left. + e^{i\theta}\frac{3Q}{4\beta_0^3}(1 - {}^2/_5\beta_0^2)\left[\xi\beta_0 - \frac{\xi\operatorname{sh} 2\beta_0}{\xi^2 + 4\beta_0^2} + (1 - {}^1/_2\xi^2)\operatorname{arctg}\frac{2\beta_0}{\xi}\right]\right\}_{\theta=0}^{\theta=\theta}$$

$$(\xi = \theta_1 - \theta) \tag{4.13}$$

with $\zeta(\tau)$ obtained from (4.7) and the constant r_2 from (4.4).

3°. When studying the flows leading to Problem B we must know not only the boundary of the stagnation zone, but also the reduced pressure drop H between the sources and sinks (regarded here as pores of small, though finite radius).

The reduced pressure can be found as a function of (u, θ) by interating the equations (see [1])

$$\frac{\partial H}{\partial u} = \frac{1+u}{u^2}\frac{\partial \psi}{\partial \theta}, \quad \frac{\partial H}{\partial \theta} = -\frac{(1+u)^2}{u}\frac{\partial \psi}{\partial u} \tag{4.14}$$

Let us use these equations to determine the pressure difference between the points on the streamline $\psi = 0$. To start with, we will choose a point $u = u_1 \gg 1$, i.e. a point in the physical plane adjacent to the source (at a distance of the order of $R_1 = Q/(\pi u_1)$). Pressure difference between this point and the tip of the stagnation zone is

$$H_1 - H_0 = \int_0^{u_1} \frac{1+u}{u^3}\frac{\partial \psi(u, 0)}{\partial \theta} du = r_2 \int_0^{\infty} \frac{(1+s^2)[1 - F_0(s, -u_1)]}{s \operatorname{th} \pi s \operatorname{sh} s\theta_1} \times$$

$$\times \int_0^{2\beta_0} \zeta(\tau) \sin s\tau d\tau ds = \frac{Q \ln u_1}{\theta_1} - \frac{Q}{\theta_1} + 0(a_0) + 0(1/\ln u_1) \tag{4.15}$$

Pressure difference between the points on the boundary of the stagnation zone is

$$H_0 - H(\theta) = \int_0^{\theta} \chi(\theta) d\theta = r_2 \int_0^{\infty} \frac{(1+s^2)(\operatorname{ch} s\theta - 1)}{s \operatorname{th} \pi s \operatorname{sh} s\theta_1} \int_0^{2\beta_0} \zeta(\tau) \sin s\tau d\tau ds$$

In particular, when $\theta = \theta_1$ we have

$$H_0 - H(\theta_1) = r_2 \int_0^{\infty} \frac{1+s^2}{s \operatorname{th} \pi s} \operatorname{th}\frac{s\theta_1}{2} \int_0^{2\beta_0} \zeta(\tau) \sin s\tau d\tau ds \approx$$

$$\approx 2Q \int_0^{\infty} (1+s^2)\left(\frac{\operatorname{th} {}^1/_2 s\theta_1}{\operatorname{th} \pi s} - 1\right) ds + \frac{3\pi Q}{8\beta_0^3} + \frac{3\pi Q}{5\beta_0} + 0(1) \tag{4.16}$$

Up to now we have been considering the part of solution corresponding to $0 \leqslant \theta \leqslant \theta_1$. But since Problem B is obviously symmetric, all the formulas remain valid also for region $\theta_1 \leqslant \theta \leqslant \theta_0$ provided that θ is replaced by $\theta_0 - \theta$ and θ by $\theta_2 = \theta_0 - \theta$, and, that the coordinate origin coincides with a sink instead of a source. We can thus construct the whole of the boundary of the stagnation zone.

4°. Example. The formulas derived enable us to determine all quantities of practical interest, relating to the sink-source problem [1] (or, which amounts to the same, to the problem of a source situated near a rectilinear delivery contour (i.e. the line of constant pressure)). For this case we have $\theta_1 = \theta_2 = \pi$.

Formula (4.12) yields the following expression for the distance between the source and the tip of the stagnation zone

$$\lambda z_0 = 2Q \int_0^{\infty} s^2 \frac{\operatorname{ch} \pi s}{\operatorname{sh}^2 \pi s} ds = \frac{Q}{\pi} \tag{4.17}$$

Further, from (4.14) we have

$$\frac{\lambda[z(\theta) - z_0]}{Q} = -e^{i\theta}\left\{\frac{4}{(\pi-\theta)^3} - \frac{\theta \sin {}^1/_2\theta}{2\pi(\cos {}^1/_2\theta)^3} - \frac{1}{\pi(\cos {}^1/_2\theta)^2} - \right.$$

$$-\frac{3}{4\beta_0^3}(1 - {}^2/_5\beta_0^2)\left[(\pi-\theta)\beta_0 - \frac{(\pi-\theta)\operatorname{sh} 2\beta_0}{(\pi-\theta)^2 + 4\beta_0^2} + (1 - {}^1/_2(\pi-\theta)^2) \operatorname{arctg}\frac{2\beta_0}{\pi-\theta}\right] +$$

$$+ i\left[\frac{1}{\pi}\operatorname{tg}\frac{\theta}{2} + \frac{\theta}{2\pi(\cos {}^1/_2\theta)^2} - \frac{2}{(\pi-\theta)^2} + \frac{3}{4\beta_0^3}(1 - {}^2/_5\beta_0^2) \times\right.$$

$$\left.\left.\times\left(2\beta_0 - {}^1/_5\beta_0(\pi-\theta)^2 + {}^4/_9\beta_0^3 - (\pi-\theta)(1 - {}^1/_6(\pi-\theta)^2) \operatorname{arctg}\frac{2\beta_0}{\pi-\theta}\right)\right]\right\} \tag{4.18}$$

Finally, (4.16) gives

$$H_0 - H(\pi) = {}^3/_8\pi\beta_0^{-3}Q + {}^3/_5\pi\beta_0^{-1}Q - {}^{10}/_3\pi^{-1}Q \qquad (4.19)$$

Setting $\theta = \pi$ in (4.18) we find

$$\lambda\,[z(\pi) - z_0]/Q = -{}^3/_8\pi\beta_0^{-3}(1 - 0.4\beta_0^2) + i\,({}^3/_2\beta_0^{-2} - 0.1) \qquad (4.20)$$

This relation connects the parameter β_0 with the half-distance between the source and the sink $L_0 = -x(\pi)$. Figure 2 depicts the boundary of the stagnation zone computed according to the formula (4.19) for several values of $a_0 = \mathrm{sh}^2\beta_0$.

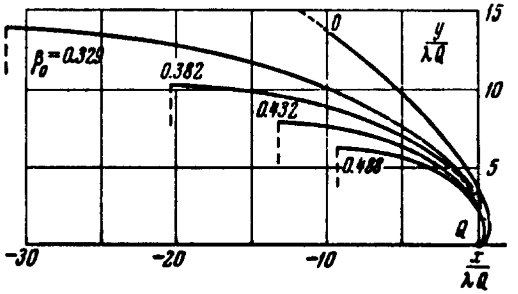

Fig. 2

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